Дз 2

№ 3

$$\begin{cases} u_t = a^2 u_{xx} & 0 < x < l \quad t > 0 \\ u(0,t) = 0 & t > 0 \\ u_x(l,t) = 0 & \\ u(x,0) = \varphi(x) & 0 \le x \le l \end{cases}$$

$u = X(x)\cdot T(t)$

$X \cdot T' = a^2 X'' T$

$\frac{X''}{X} = \frac{T'}{a^2 T} = -\lambda$

I $\quad X'' = -\lambda X$

1) $\lambda = 0$

$\begin{cases} X'' = 0 \\ X(0) = 0 \\ X'(l) = 0 \end{cases}$

$\Rightarrow X = ax + b$

$X(0) = b = 0$

$X'(\ ) = a = 0$

$\Rightarrow X = 0$

2) $\lambda < 0, \ -\lambda = \mu^2$

$\begin{cases} X'' = \mu^2 X \\ X(0) = 0 \\ X'(l) = 0 \end{cases}$

$\Rightarrow X = ae^{-\mu x} + be^{\mu x}$

$X(0) = a + b = 0$

$X'(l) = -ae^{-\mu l}\mu + b\mu e^{\mu l} = 0$

$a = -b$

$be^{-\mu} + be^{\mu l} = 0$

$\Rightarrow b = 0$

$\Rightarrow X = 0$

3) $\lambda > 0 \quad \lambda = \mu^2$

$\begin{cases} X'' = -\mu^2 X \\ X(0) = 0 \\ X'(\ ) = 0 \end{cases}$

$= X = a \sin \mu x + b \cos \mu x$

$X(0) = b = 0$

$X'(l) = a\mu \cos(\mu x) = 0$

$\to X = 0$

$\to \mu \cdot l = \frac{\pi}{2} + \pi n \quad n \in \mathbb{N} \cup \{0\}$, т.к $\mu > 0$

$\Rightarrow X_n = a_n \sin(\sqrt{\lambda_n}\, x)$

$\sqrt{\lambda_n} = \mu_n = \frac{\left(\frac{\pi}{2} + \pi n\right)}{l} \quad n \in \mathbb{N} \cup \{0\}$

$\lambda_n = \left(\frac{\pi + 2\pi n}{2l}\right)^2 \quad n = 0, 1, \ldots$

II $\quad T' = -\lambda_n a^2 T$

$T_n = b_n e^{-\lambda_n a^2 t}$

III $\quad u(x,t) = \sum_{n=0}^{\infty} c_n \sin(\sqrt{\lambda_n}\, x)\, e^{-\lambda_n a^2 t}$

$u(x,0) = \sum_{n=0}^{\infty} c_n \sin(\sqrt{\lambda_n}\, x) = \varphi(x)$

$\Rightarrow c_n = \frac{(\varphi(x), \sin\sqrt{\lambda_n}\, x)}{\|X_n\|^2} = \frac{1}{\|\sin\sqrt{\lambda_n}\, x\|^2} \int_0^l \varphi(x) \sin\sqrt{\lambda_n}\, x \, dx$

$\|\sin\sqrt{\lambda_n}\, x\|^2 = \frac{l}{2}$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{\pi + 2\pi n}{2l} x\right) dx \, \sin\left(\frac{\pi + 2\pi n}{2l} x\right) e^{-\left(\frac{\pi + 2\pi n}{2l}\right)^2 a^2 t}}$$

№ 5 Дз 2

$$\begin{cases} u_t = u_{xx} & 0<x<1 \quad t>0 \\ u_x(0,t)=0 & u(\,,t)=0 \quad t>0 \\ u(x,0)=x^2-1 & 0\le x\le l \end{cases}$$

$$u(x,t)=X(x)\cdot T(t)$$

$$X_n=\cos(\sqrt{\lambda_n}x); \quad \lambda_n=\left(\frac{\pi+2\pi n}{2}\right)^2 \quad T_n=e^{-\lambda_n a^2 t} \qquad n=0,1,..$$

$$u(x,t)=\sum_{n=0}^{\infty} C_n\cos(\sqrt{\lambda_n}x)\,e^{-\lambda_n a^2 t}$$

$$C_n=\frac{1}{\|X_n\|^2}\int_0^1 X_n(x)\cdot\varphi(x)\,dx$$

$$\|X_n\|^2=\frac{l}{2}=\frac{1}{2}$$

$$\int_0^1\cos(\sqrt{\lambda_n}x)(x^2-1)\,dx=-\int_0^1\cos x\left(\frac{\pi}{2}+\pi n\right)dx+\int_0^1 x^2\cos\left(\left(\frac{\pi}{2}+\pi n\right)x\right)dx=-\frac{\sin\left(\frac{\pi}{2}+\pi n\right)x}{\frac{\pi}{2}+\pi n}\Bigg|_0^1+$$

$$+\frac{x^2\sin\left(\frac{\pi}{2}+\pi n\right)x}{\frac{\pi}{2}+\pi n}\Bigg|_0^1-\int_0^1\frac{2x\sin\left(\frac{\pi}{2}+\pi n\right)x}{\frac{\pi}{2}+\pi n}dx=-\frac{\sin\left(\frac{\pi}{2}+\pi n\right)}{\frac{\pi}{2}+\pi n}+\frac{\sin\left(\frac{\pi}{2}+\pi n\right)}{\frac{\pi}{2}+\pi n}$$

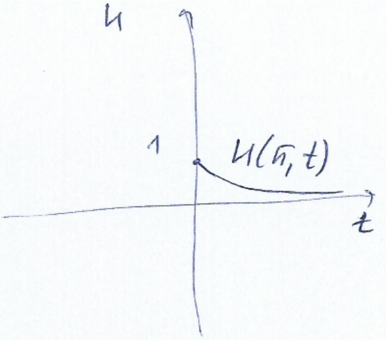
$$+\frac{2x\cos\left(\frac{\pi}{2}+\pi n\right)x}{\left(\frac{\pi}{2}+\pi n\right)^2}\Bigg|_0^1-\int_0^1\frac{2\cos\left(\frac{\pi}{2}+\pi n\right)x}{\left(\frac{\pi}{2}+\pi n\right)^2}dx=-\frac{2\sin\left(\frac{\pi}{2}+\pi n\right)x}{\left(\frac{\pi}{2}+\pi n\right)^3}\Bigg|_0^1=\frac{-2(-1)^n}{\left(\frac{\pi}{2}+\pi n\right)^3}=$$

$$=\frac{+2(-1)^{n+1}}{\left(\frac{\pi}{2}+\pi n\right)^3}$$

$$\boxed{u(x,t)=\sum_{n=0}^{\infty}\frac{4(-1)^{n+1}}{\left(\frac{\pi}{2}+\pi n\right)^3}\cos\left(\frac{\pi+2\pi n}{2}x\right)e^{-\left(\frac{\pi+2\pi n}{2}\right)^2 t}}$$

Дз 2

№9



$$\begin{cases} u_t = a^2 u_{xx} & 0 < x < \pi \quad t > 0 \\ u(0,t) = 0 & t > 0 \\ u_x(\pi, t) = 0 & t > 0 \\ u(x,0) = \sin\frac{5x}{2} & 0 \le x \le \pi \end{cases}$$

$X_n = \sin(\sqrt{\lambda_n}\, x) \qquad \lambda_n = \left(\frac{\pi + 2\pi n}{2l}\right)^2 = \left(\frac{1+2n}{2}\right)^2 \qquad T_n = e^{-\lambda_n a^2 t}$

$u(x,t) = \sum\limits_{n=0}^{\infty} c_n \sin(\sqrt{\lambda_n}\, x) e^{-\lambda_n a^2 t}$

$u(x,0) = \sum\limits_{n=0}^{\infty} C_n \sin\left(\frac{1+2n}{2} x\right) = \sin\frac{5x}{2}$

$\Rightarrow \frac{1+2n}{2} = \frac{5}{2} \Rightarrow n = 2 \Rightarrow C_2 = 1, \quad C_i = 0 \quad i \in \mathbb{N} \cup \{0\} / \{2\}$

$\boxed{u(x,t) = \sin\frac{5x}{2} \cdot e^{-\frac{25}{4} a^2 t}}, \quad \boxed{\lim\limits_{t \to \infty} u(x,t) = 0}, \quad u(\pi, t) = \sin\frac{5\pi}{2} e^{-\frac{25}{4} a^2 t} = e^{-\frac{25}{4} a^2 t}$

№12

$$\begin{cases} u_t = a^2 \cdot u_{xx} - b(u - U), & 0 < x < l \quad t > 0 \\ \qquad b = const > 0 \qquad U - const \\ u_x(0,t) = 0 & t > 0 \\ u(l, t) = U & t > 0 \\ u(x,0) = u_0 & 0 \le x \le l \end{cases}$$

Замена $u(x,t) = e^{-bt} v + U$

$-b e^{-bt} v + e^{-bt} v_t = a^2 e^{-bt} v_{xx} - b(e^{-bt} v + U - U) \quad | : e^{-bt} > 0$

$$\begin{cases} v_t = a^2 v_{xx} \\ v_x(0,t) = 0 \\ v(l,t) = 0 \\ v(x,0) = u_0 - U = \varphi_0 \end{cases}$$

$X_n(x) = \cos(\sqrt{\lambda_n}\, x) \qquad \lambda_n = \left(\frac{\pi + 2\pi n}{2l}\right)^2 \qquad T_n = e^{-\lambda_n a^2 t}$

$v(x,t) = \sum\limits_{n=0}^{\infty} \cos(\sqrt{\lambda_n}\, x) \cdot e^{-\lambda_n a^2 t} \cdot C_n$

$v(x,0) = \sum\limits_{n=0}^{\infty} C_n \cos(\sqrt{\lambda_n}\, x) = \varphi_0$

$C_n = \frac{2}{l} \int\limits_0^l \varphi_0 \cos(\sqrt{\lambda_n}\, x) dx = \frac{2}{l} \int\limits_0^l \varphi_0 \cos\left(\frac{\pi + 2\pi n}{2l} x\right) dx = \frac{2\varphi_0}{l} \left. \frac{\sin\left(\frac{\pi + 2\pi n}{2l} x\right)}{\frac{\pi + 2\pi n}{2l}} \right|_0^l =$

$= \frac{2\varphi_0}{l} \frac{\sin\left(\frac{\pi}{2} + \pi n\right)}{\frac{\pi + 2\pi n}{2l}} = \frac{4\varphi_0 (-1)^n}{\pi + 2\pi n}$

$\boxed{u(x,t) = U + e^{-bt} \left(\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{\pi + 2\pi n} \cos\left(\frac{\pi + 2\pi n}{2l} x\right) \cdot e^{-\left(\frac{\pi + 2\pi n}{2l} a\right)^2 t} \right) \cdot 4(u_0 - U)}$

$\boxed{\lim\limits_{t \to \infty} u(x,t) = U}$, условие $b > 0$ существенное для сх-ти